# Intra-session coding

In intra-session network coding, symbols (sometimes also called blocks of data) from the same file or stream are combined to create new coded symbols.

Consider that we have some data and that we arrange it in the matrix $\boldsymbol{M}$. Each of the columns in the matrix corresponds to a symbol. In order to create a coded symbol $\boldsymbol{x}$ we combine the symbols as per defined by a coding vector, $\boldsymbol{g}$.

$$\boldsymbol{M}_{b\times g} \cdot \boldsymbol{g}_g = \boldsymbol{x}_b \tag{1}$$

In this way we can generate an unlimited number of coded packets, by drawing the coding vector $\boldsymbol{g}$ at random. We can write this as the following, where the columns in $\boldsymbol{G}$ corresponds to coding vectors.

$$\boldsymbol{M}_{b\times g} \cdot \boldsymbol{G}_{g\times r} = \boldsymbol{X}_{b\times r} \tag{2}$$

Thus a node the receives coded symbols can decode the original data in the following way.

$$\boldsymbol{X} \cdot \boldsymbol{G}^{-1} = \boldsymbol{M} \tag{3}$$

The requirement is, however, that $\boldsymbol{G}$ is invertible, which is when $\boldsymbol{G}$ is full rank. Currently, all computer systems have finite precision and thus to avoid rounding errors, all coding operation are performed over a finite field. In this this exercise, we will use the binary field ($q = 2$), which is easy to compute by hand.

**Exercise 1: Encode and decode** ..........................................................................

(a) What is the lowest reasonable value of $r$ and why?

**Solution:** $r \geq g$ as otherwise $\boldsymbol{G}$ will not be invertible.

(b) Given the following $\boldsymbol{M}$ and $\boldsymbol{G}$ encode the symbols in $\boldsymbol{X}$.

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{G} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \tag{4}$$

**Solution:**

$$\boldsymbol{X} = \boldsymbol{M} \cdot \boldsymbol{G} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \tag{5}$$

Now lets assume that the symbols are transmitted over a network and arriving at a receiver, which receives $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{G}}$, the hat denotes that not necessarily all symbols sent by the source are received by the sink.

$$\left[\ \hat{\boldsymbol{G}}\ \middle|\ \boldsymbol{I}\ \right] \rightarrow \left[\ \boldsymbol{I}\ \middle|\ \hat{\boldsymbol{G}}^{-1}\ \right] \tag{6}$$

$$\hat{\boldsymbol{X}} \cdot \hat{\boldsymbol{G}}^{-1} = \hat{\boldsymbol{M}} \tag{7}$$

$$\tag{8}$$

(c) Assuming $\hat{\boldsymbol{G}} = \boldsymbol{G}$ and $\hat{\boldsymbol{X}} = \boldsymbol{X}$ find $\hat{\boldsymbol{M}} = \boldsymbol{M}$.

**Solution:**

$$\hat{\boldsymbol{G}}^{-1} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{9}$$

$$\boldsymbol{M} = \hat{\boldsymbol{M}} = \hat{\boldsymbol{X}} \cdot \hat{\boldsymbol{G}}^{-1} \tag{10}$$

We have now seen that we can encode data and decode it again. Consider why this might be useful in the following cases.

(d) If there are erasures in the network, and the source wants to repair those erasures, what should the source do?

**Solution:** The source can simply generate new symbols and transmit these.

(e) If the source does not know how many erasures occurred in the network, what should it do?

**Solution:** The source can generate simply keep transmitting coded symbols until the sink notifies that it has successfully decoded.

(f) Why might it be useful to be able to produce an unlimited number of new coded symbols? (codes with this property are called rateless)

**Solution:** Because any erasure rate in the network can be overcome.

**Exercise 2: Decoding probability and Overhead** ..................................................

In the last exercise we decoded the original data, however it is not guaranteed that a sink can always decode. Consider the following coding matrices where the name indicates the rank of the matrix

$$\boldsymbol{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

(a) For each matrix determine what coded symbols increase the rank and which does not (there are 16 possible coding vectors).

**Solution:** Write up the possible coding vectors - there are 16 permutations. For each of the five matrices determine how many of the 16 vectors are linear independent - the rest are linear dependent.

(b) Based on this, define $P_{r \to r}$, that is the probability that any randomly coded incoming symbol does not increase the rank of the coding matrix.

> **Solution:** Based on the previous exercise we can deduce that:
>
> $$P_{r \to r} = \frac{1}{2^{g-r}} \tag{12}$$

(c) Instead of $q = 2$ assume that $q$ can take any value and define $P_{r \to r}$.

> **Solution:**
>
> $$P_{r \to r} = \frac{1}{q^{g-r}} \tag{13}$$

(d) Define the expression $O_r$ which is the expected overhead for successfully increasing the rank at a receiver when a symbol is received, given that the rank is $r$ and the size of the decoding matrix is $g$.

> **Solution:**
>
> $$O_r = \sum_{i=0}^{\infty} i \left(P_{r \to r}\right)^{i-1} \left(1 - P_{r \to r}\right) \tag{14}$$
>
> $$= \frac{1}{\left(1 - P_{r \to r}\right)^2} \left(1 - P_{r \to r}\right) \tag{15}$$
>
> $$= \frac{1}{\left(1 - P_{r \to r}\right)} \tag{16}$$
>
> $$= \frac{1}{1 - \frac{1}{q^{g-r}}} \tag{17}$$

(e) Based on $O_r$ calculate the expected overhead for a matrix of size $g$ to attain full rank.

> **Solution:**
>
> $$O_T = \sum_{r=0}^{g-1} O_r \tag{18}$$
>
> $$= \frac{1}{1 - \frac{1}{q^{g-r}}} \tag{19}$$
>
> $$= \sum_{r'=1}^{g} \left( \frac{1}{1 - q^{-r'}} \right) \tag{20}$$

(f) Calculate the overhead for $q = [2, 2^8, 2^{32}]$ and $g = [8, 64, 1024]$.

> **Solution:**
> For $q = 2$ the overhead is approximately 1.6.
> For higher q the overhead is negligible.