Finite Fields

In practice most finite field applications e.g. cryptography and error correcting codes utilizes a specific type of finite fields, namely the *binary extension fields*. The following exercises will introduce you to calculations in binary extension fields.

Exercise 1: The Binary Field ....

The binary field $\mathbb{F}_2$ consists of two elements $\{0, 1\}$ and is of particular interest since the binary operations are easily implemented and represented in software and hardware. In the case of the binary fields, arithmetic operations are performed modulo-2. For addition and multiplication this corresponds to the bitwise exclusive or (XOR) and the bitwise and (AND) operations.

(a) Fill in the missing values in the below table.

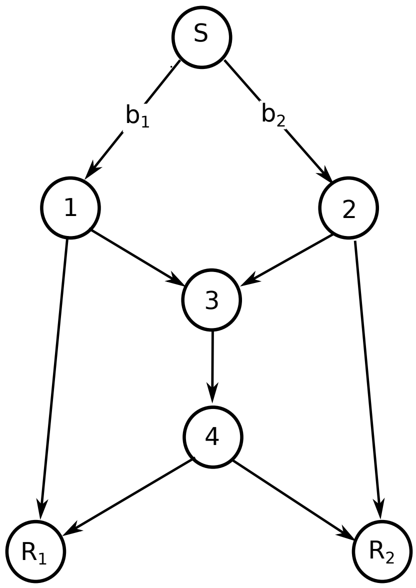
XOR	0	1
0		
1		

AND	0	1
0		
1		

Table 1: The finite field $\mathbb{F}_2$ consists of elements 0 and 1 which satisfy the addition(XOR) and multiplication(AND) tables.

Exercise 2: The Butterfly Network ....

(a) The butterfly network shown in the below figure is a famous example of Network Coding. Show how the two bits b_1 and b_2 may be delivered at the two receivers R_1 and R_2 simultaneously using the operations from the binary field (hint: the XOR operation is enough). Each edge may carry 1 bit per time unit.



Exercise 3: Binary Field Calculations ....

In practice calculations are typically performed w-bits at a time, where w corresponds to the word size of the hardware used. Typical values of w are $\{16, 32, 64\}$ bits. The below table shows an example of XOR and AND between two 8-bit words.

$$\begin{array}{rl} & 01011001 \\ XOR & 00100110 \\ = & 01111111 \end{array} \qquad \begin{array}{rl} & 00011101 \\ AND & 01100111 \\ = & 00000101 \end{array}$$

	11101001			01111000
XOR	10100110		AND	01100111
$=$			$=$	

Table 2: Addition and multiplication in $\mathbb{F}_2$ with 8-bit words.

(a) Fill in the missing values in the tables.

Exercise 4: The Binary Extension Field ....

In a previous exercise we found a solution for the butterfly network using the binary field. However, when implementing Network Coding it can in certain cases be necessary to increase the field size in order to find a solution (i.e. the number of elements contained in the field). In the binary field $\mathbb{F}_2$ the field size was 2, i.e. there are two elements $0, 1$. Binary extension fields have the form $\mathbb{F}_{2^m}$, where $m \geq 1$. A binary extension field contains 2^m elements, all field elements may be represented as binary polynomials of degree at most $m-1$:

$$\mathbb{F}_{2^m} = \{f_{m-1}x^{m-1} + \ldots + f_2x^2 + f_1x + f_0 : f_i \in \{0, 1\}\} \tag{1}$$

As an example consider the field given by $\mathbb{F}_{2^3}$ in the below table, in this case the field will consist of $2^3 = 8$ polynomial elements of degree $< m$.

In the binary extension field all polynomial elements can be represented as m bit binary numbers. It is important to notice the correspondence between the binary and polynomial representation. The bits from left to right are the coefficients of the powers of x in increasing order.

$GF2^3$		
Polynomial	Binary	Decimal
0	000	0
1	001	1
x	010	2
$x+1$	011	3
x^2	100	4
x^2+1	101	5
x^2+x	110	6
x^2+x+1	111	7

(a) Fill in the missing values in the below table.

$GF2^m$		
Polynomial	Binary	Decimal
$x^7+x^6+x^4+x+1$		
	11001001	
		133
x^4+x^2+x		
	00011001	
		10

(b) In the table what is the required value for m in order to represent the field elements.

Exercise 5: Polynomial Addition and Subtraction ....

Polynomials essentially allow the same arithmetic operations as integers, however when polynomials are used the operations are performed modulo-$p(x)$, where $p(x)$ is an irreducible polynomial instead of a prime integer (e.g. 2 as in the case of the binary field). As with a prime number an irreducible polynomial is one which cannot be factored into products of two polynomials.

Ordinary polynomial addition is performed component-wise e.g. for two polynomials with a maximum degree of k:

$$f(x) = h(x) + g(x) \tag{2}$$

$$f(x) = \sum_{i=0}^{k} (h_i + g_i)x^i \tag{3}$$

In $\mathbb{F}_{p^m}$ we calculate $f(x)+g(x)$ as $f(x)+g(x) \mod p(x)$. This uses the usual component-wise addition as given in Equation (3), the only difference is that the coefficient sum is modulo p i.e. $h_i + g_i \mod p$. As the degree of the resulting polynomial $f(x)$ cannot exceed the degree of the chosen prime polynomial, no further computations are needed.

Example

Lets consider a $w = 8$ bit architecture with the two polynomials $a(x) = x^7 + x^6 + x^2$ and $b(x) = x^7+x^5+x^3+x^2$ with the binary representation of 11000100 and 10101100 respectively. In the following we use $\oplus$ to denote the XOR operation.

- Addition or subtraction: $11000100 \oplus 10101100 = 01101000$.

The result may be confirmed by adding the two polynomials directly:

$$\begin{aligned} f(x) &= (x^7 + x^6 + x^2) + (x^7 + x^5 + x^3 + x^2) && (4)\\ &= (1 \oplus 1)x^7 + x^6 + x^5 + x^3 + (1 \oplus 1)x^2 && (5)\\ &= x^6 + x^5 + x^3 && (6) \end{aligned}$$

Where $x^6 + x^5 + x^3$ has the expected binary representation 01101000.

(a) In $\mathbb{F}_{2^8}$ calculate $(x^5 + x) + (x^3 + x^2)$

(b) In $\mathbb{F}_{2^8}$ calculate $(x^7 + x^3) + (x^7 + x + 1)$

(c) In $\mathbb{F}_{2^8}$ calculate $(x^3 + x^2 + x + 1) + (x + 1)$

(d) In $\mathbb{F}_{2^8}$ calculate $(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 1) + (x^4 + x^2 + 1)$
In the binary and binary extension field subtraction and addition are identical (based on the XOR).

Subtraction of two field elements can be defined in terms of addition, if $a, b \in \mathbb{F}$ then $a - b = a + (-b)$, where $-b$ is the unique field element in $\mathbb{F}$ such that $b + (-b) = 0$ ($-b$ is called the negative of b).

(e) In $\mathbb{F}_{2^8}$ calculate $(x^5 + x) - (x^3 + x^2)$

(f) In $\mathbb{F}_{2^8}$ calculate $(x^7 + x^3) - (x^7 + x + 1)$

(g) In $\mathbb{F}_{2^8}$ calculate $(x^3 + x^2 + x + 1) - (x + 1)$

(h) In $\mathbb{F}_{2^8}$ calculate $(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 1) - (x^4 + x^2 + 1)$

Exercise 6: Polynomial Multiplication ....

For ordinary polynomial multiplication, the coefficients of $f(x) = h(x)g(x)$ are determined by convolution, the resulting polynomial $f(x)$ is of degree = deg(h)+deg(g):

$$f_i = \sum_{j=0}^{i} h_j g_{i-j} \tag{7}$$

The product $h(x)g(x)$ in $\mathbb{F}_{p^m}$ can be found by first multiplying $h(x)$ and $g(x)$ using ordinary polynomial multiplication. Then ensuring that the resulting polynomial $f(x)$ has degree $< m$ by reducing it modulo $p(x)$. The modulo operation can be implemented as polynomial long division and then taking the remainder. As for polynomial addition we must also ensure that all resulting coefficients are elements in $\mathbb{F}_p$ by reducing them modulo p.

In the following use the irreducible prime polynomial $p(x) = x^5 + x^2 + 1$

Example

$$\begin{aligned} f(x) &= (x^3 + x + 1) \cdot (x^2 + 1) && (8)\\ &= x^5 + (1 \oplus 1)x^3 + x^2 + x + 1 && (9)\\ &= x^5 + x^2 + x + 1 && (10)\end{aligned}$$

Since degree of $f(x)$ equal m, we perform the modulo operation using the irreducible polynomial $p(x)$

$$\begin{aligned} f(x) &= x^5 + x^2 + x + 1 \mod p(x) && (11)\\ f(x) &= x && (12)\end{aligned}$$

The result of the modulo operation can be calculated as the remaineder of polynomial long division with the irreducible polynomial:

$$\begin{array}{r|rrrrl} & & & & 1 & \triangleleft \text{ qoutient}\\ \cline{2-5} x^5 + x^2 + 1 & x^5 & +x^2 & +x & +1 & \\ & x^5 & +x^2 & & +1 & \\ \cline{2-5} & & & x & & \triangleleft \text{ remainder}\\ \cline{4-5} \end{array}$$

(a) In $\mathbb{F}_{2^5}$ calculate $(x^4 + x) \cdot (x^3 + x^2)$

(b) In $\mathbb{F}_{2^5}$ calculate $(x^3) \cdot (x^2 + x^1 + 1)$

Exercise 7: Polynomial Division ....

Division can be implemented in terms of multiplication with the inverse element.

Division can be defined in terms of multiplication: if $a, b \in \mathbb{F}$ then $a/b = a \cdot (b^{-1})$, where b^{-1} is the unique field element in $\mathbb{F}$ such that $b \cdot b^{-1} = 1$ (b^{-1} is called the inverse of b).

The inverse of a polynomial may be found using the Extended Euclidean algorithm.

(a) In $\mathbb{F}_{2^5}$ calculate $(x^4 + x)/(x^3 + x^2)$ given $(x^3 + x^2)^{-1} = (x^2 + x + 1)$

(b) In $\mathbb{F}_{2^5}$ verify $(x^3 + x^2)^{-1} = (x^2 + x + 1)$